Analyzing Boninger's result on $\mathbb{T}$

All knot diagrams are assumed to be checkerboard colorable.

When looking at knot diagrams on $\mathbb{T}^m$, their Tait graphs can be seen as <u>generalized</u> <u>ribbon</u> <u>graphs</u>, whose vertices may be more complicated than just discs.

<u>Def.</u> Let D be a knot diagram on $\mathbb{T}^m$ with generalized Tait graph Γ. We define

$$h(\Gamma) = \sum_{v \in V(\Gamma)} bc(v) - |V(\Gamma)|,$$

where $bc(v)$ denotes the number of boundary components of vertex v.

$h(\Gamma)$ can be thought of as the total number of holes in the vertices of Γ. $h(\Gamma) = 0$ iff all the vertices are discs.

Let D be a knot diagram on $\mathbb{T}$ with generalized Tait graphs Γ_1, Γ_2, and assume $h(\Gamma_1) \leq h(\Gamma_2)$. Then either

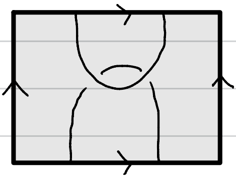
(1) $h(\Gamma_1) = h(\Gamma_2) = 0$, or

(2) $h(\Gamma_1) = 0$, $h(\Gamma_2) = 1$.

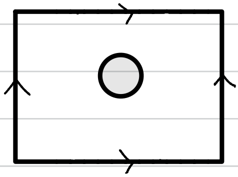
More generally, if D is a knot diagram on $\mathbb{T}^m$ with generalized Tait graphs Γ_1, Γ_2, then $h(\Gamma_1) + h(\Gamma_2) \leq m$. (?)

Reduced diagrams

Let D be a Knot diagram on $\mathbb{T}$. We say that D is reduced if it can't be "Simplified."

Eg. 

This diagram is not reduced, because it is equivalent to the following, simpler diagram:



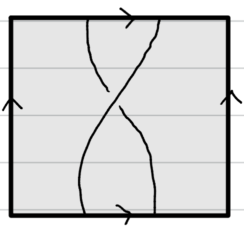
So, D is reduced if it satisfies one of the following:

(1) D can be drawn on a plane.

(2) D can be drawn on a cylinder but D is not equivalent (on the torus) to a diagram which can be drawn on the plane.

(3) D is not equivalent (on the torus) to a diagram which can be drawn on a cylinder.

reduced diagrams:



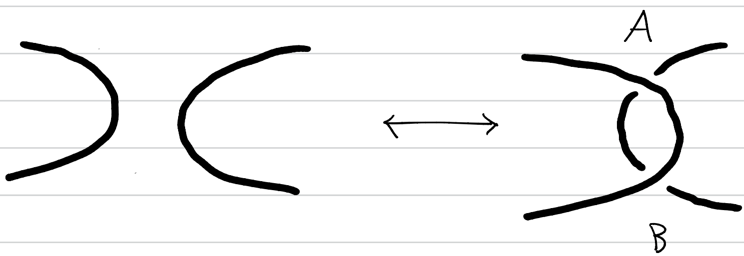
If D is a diagram on $\mathbb{T}$ equivalent to reduced diagrams D_1, D_2, then $h(D_1) = h(D_2)$.

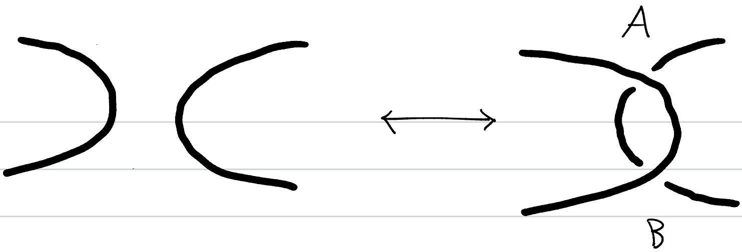
<u>Claim</u>. Let D be a knot diagram on Π with generalized Tait graphs Γ_1, Γ_2. Let D' be a reduced diagram equivalent to D with corresponding Tait graphs Γ_1', Γ_2'. Then the following is invariant under Reidemeister moves:

$$\left\{ (-A^2-A^{-2})^{\pm|h(\Gamma_1)-h(\Gamma_1')|}\, \nu_{D,\Gamma_1},\ (-A^2-A^{-2})^{\pm|h(\Gamma_2)-h(\Gamma_2')|}\, \nu_{D,\Gamma_2} \right\}$$

In other words, the polynomial ν always changes by a factor of $(-A^2-A^{-2})$.

<u>Proof</u>. By our previous work, we know that $\{\nu_{D,\Gamma_1}, \nu_{D,\Gamma_2}\}$ is invariant under Reidemeister moves I + III, and also move II when regions A and B are distinct, as pictured below:

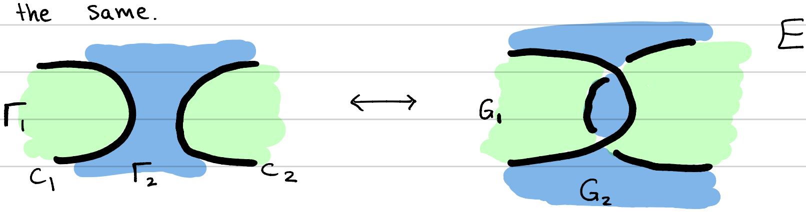




Additionally, in these cases the factor

$$(-A^2 - A^{-2})^{-|h(\Gamma_1) - h(\Gamma_1')|}$$

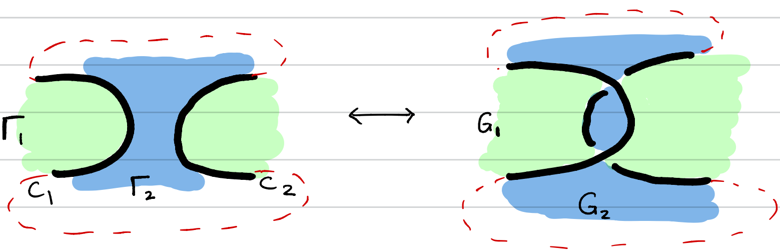
remains unchanged. So, the only case we need to check is RII as pictured above when regions A and B are the same.

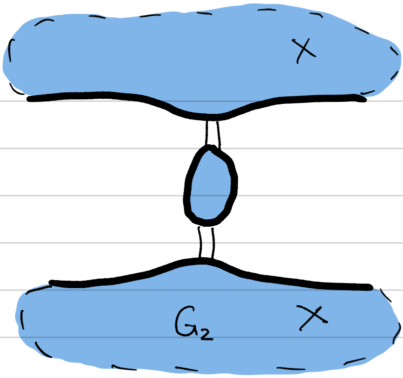


Call the new diagram E with Tait graphs G_1, G_2.
It is not hard to show that $(-A^2 - A^{-2})^{-|h(\Gamma_1) - h(\Gamma_1')|} \gamma_{D,\Gamma_1}$ is unchanged, So we only need to check the blue region.

It is also not hard to check that $\gamma_{E,G_2} = (-A^2 - A^{-2}) \gamma_{D,\Gamma_2}$.

<u>Case 1</u>. c_1, c_2 are the same curve.





$bc(G_2) = bc(\Gamma_2) + 2$, $|V(G_2)| = |V(\Gamma_2)| + 1$, so

$h(G_2) = bc(G_2) - |V(G_2)| = bc(\Gamma_2) + 2 - |V(\Gamma_2)| - 1 = (bc(\Gamma_2) - |V(\Gamma_2)|) + 1 = h(\Gamma_2) + 1.$

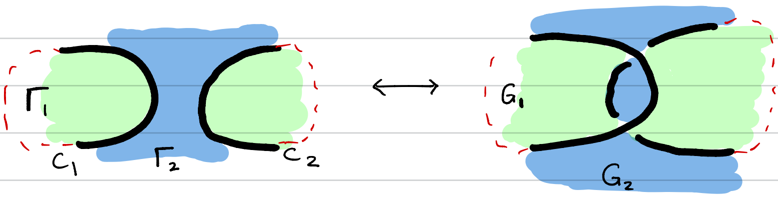
This means that $h(\Gamma_2) = 0$, $h(G_2) = 1$. This must mean that $h(\Gamma_2') = 0$.

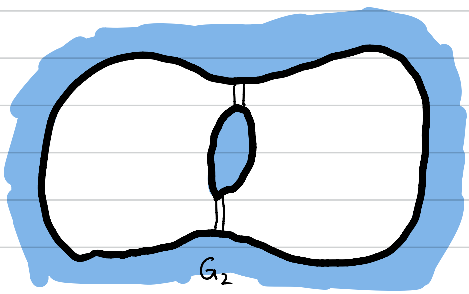
Then

$$(-A^2 - A^{-2})^{-|h(G_2) - h(\Gamma_2')|} \mathcal{V}_{E, G_2} = (-A^2 - A^{-2})^{-1} (-A^2 - A^{-2}) \mathcal{V}_{D, \Gamma_2}$$

$$= (-A^2 - A^{-2})^{-|h(\Gamma_2) - h(\Gamma_2')|} \mathcal{V}_{D, \Gamma_2}$$

<u>Case 2</u>. c_1, c_2 are distinct curves.





$bc(G_2) = bc(\Gamma_2)$, $|V(G_2)| = |V(\Gamma_2)| + 1$, so $h(G_2) = bc(G_2) - |V(G_2)|$ $= bc(\Gamma_2) - |V(\Gamma_2)| - 1 = h(\Gamma_2) - 1$. It follows that $h(\Gamma_2) = 1$, $h(G_2) = 0$, and $h(\Gamma_2') = 0$. Then

$$(-A^2 - A^{-2})^{-|h(G_2) - h(\Gamma_2')|} \mathcal{V}_{E, G_2} = (-A^2 - A^{-2})^0 \, (-A^2 - A^{-2}) \, \mathcal{V}_{D, \Gamma_2}$$

$$= (-A^2 - A^{-2})^{|h(\Gamma_2) - h(\Gamma_2')|} \mathcal{V}_{D, \Gamma_2}$$